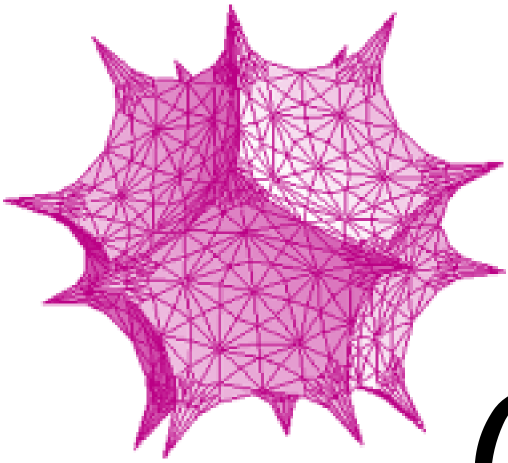

A brief history of NON-EUCLIDEAN GEOMETRY

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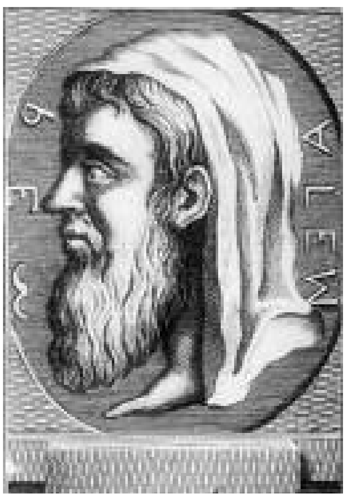
Euclid

Around 300 BC, Euclid wrote *The Elements*, a major treatise on the geometry of the time, and what would be considered 'geometry' for many years after. Arguably *The Elements* is the second most read book of the western world, falling short only to *The Bible*. In his book, Euclid states five postulates of geometry which he uses as the foundation for all his proofs. It is from these postulates we get the term *Euclidean geometry*, for in these Euclid strove to define what constitutes 'flat-surface' geometry. These postulates are:

1. [It is possible] to draw a straight line from any point to any other.
2. [It is possible] to produce a finite straight line continuously in a straight line.
3. [It is possible] to describe a circle with any centre and distance [radius].
4. That all right angles are equal to each other.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.



Euclid

It is clear that the fifth postulate is very different to the other four. In fact, in *The Elements*, the first 28 results are proved without it. As a result of this difference, many attempts were made to try to prove the fifth postulate using the previous four postulates. One earlier attempt at this was made by Proclus (410–485). Despite his attempts eventually resulting in failure, Proclus discovered an equivalent statement for the fifth postulate. This is now known as *Playfair's Axiom*. It says the following:

> Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.

Saccheri

The attempts to try and prove the fifth postulate in terms of the other four continued. The first major breakthrough was due to Girolamo Saccheri in 1697. His technique involves assuming the fifth postulate false and attempting to derive a contradiction. What Saccheri finds is shown in the diagram on page 3: the summit angles ADC and BCD are equal. This gives three cases for him to consider:

1. The summit angles are > 90 degrees (hypothesis of the obtuse angle).
2. The summit angles are < 90 degrees (hypothesis of the acute angle).
3. The summit angles are = 90 degrees (hypothesis of the right angle).

Using Euclid's assumption that a straight line is infinite, Saccheri manages to derive a contradiction for the first hypothesis and a hazy contradiction for the second one. Around 100 years later, Legendre also worked at the problem. He gives another equivalent statement to the fifth postulate, that is:

> The sum of the angles of a triangle is equal to two right angles.

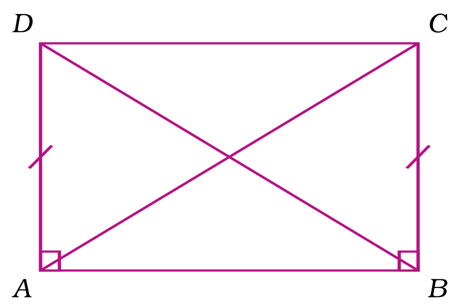
Using a similar idea to Saccheri's, Legendre showed that the sum of the angles of a triangle cannot be greater than two right angles; however his proof rests on the assumption of infinite lines. Legendre also provided a proof on the sum not being less than two right angles, but again there was a flaw, in that he makes an assumption equivalent to the fifth postulate.

Gauss and Bolyai

The first person to understand the problem of the fifth postulate was Gauss. In 1817, after looking at the problem for many years, he had become convinced it was independent of the other four. Gauss then began to look at the consequences of a geometry where this fifth postulate was not necessarily true. He never published his work due to pressure of time, perhaps illustrating Kant's statement that Euclidean geometry requires the inevitable necessity of thought. As is often the case in mathematics, similar ideas were developed independently by Janos Bolyai. His father, Wolfgang Bolyai, friend of Gauss, had once told Janos,

> You ought not to try the road of the parallels; I know the road to its end — I have passed through this bottomless night, every light and every joy of my life has been extinguished by it — I implore you for God's sake, leave the lesson of the parallels in peace… I had purposed to sacrifice myself to the truth; I would have been prepared to be a martyr if only I could have delivered to the human race a geometry cleansed of this blot. I have performed dreadful, enormous labours; I have accomplished far more than was accomplished up until now; but never have I found complete satisfaction… When I discovered that the bottom of this night cannot be reached from the earth, I turned back without solace, pitying myself and the entire human race.

Janos ignored his father's impassioned plea, however, and worked on the problem himself. Like Gauss, he looked at the consequences of the fifth postulate not being necessary. His major breakthrough, was not his work, which had already been done by Gauss, but the fact that he believed that this 'other' geometry actually existed. Despite the revolutionary new ideas that were being put forward, there was little public recognition to be had.



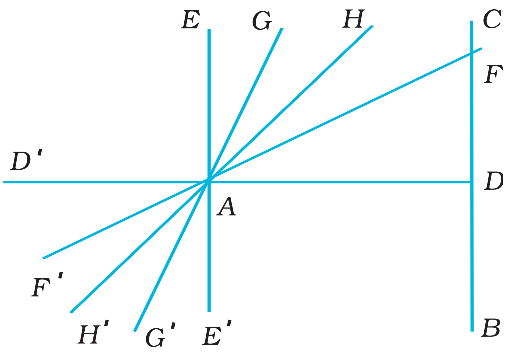
ΔABD is congruent to ΔBAC (two sides and included angle). Hence $AC = BD$ so ΔADC is congruent to ΔBCD (three sides). Therefore $\angle ADC = \angle BCD$.

Saccheri's quadrilateral

Lobachevsky

Another mathematician, Lobachevsky, worked on the same problems as Gauss and Bolyai but again, despite working at the same time, he knew nothing of their work. Lobachevsky also assumed the fifth postulate was not necessary and from this formed a new geometry. In 1840, he explained how this new geometry would work (see diagram on page 4):

> All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes — into cutting and non-cutting. The



AD is the perpendicular from *A* to *BC*.
AE is perpendicular to *AD*.
Within the angle *EAD*, some lines (such as *AF*) will meet *BC*. Assume that *AE* is not the only line which does not meet *BC*, so let *AG* be another such line.
AF is a cutting line and *AG* is a non-cutting line. There must be a boundary between cutting and non-cutting lines and we may take *AH* as this boundary.

Part of Lobachevsky's calculation.

> boundary lines of the one and the other class of those lines will be called parallel to the given line.

From this, Lobachevsky's geometry has a new fifth postulate, that is:

> There exist two lines parallel to a given line through a given point not on the line.

Clearly, this is not equivalent to Euclid's geometry. Lobachevsky went on to develop many trigonometric identities for triangles which held in this geometry, showing that as the triangle becomes small the identities tend to the usual trigonometric identities.

Felix Klein
(1849–1925)

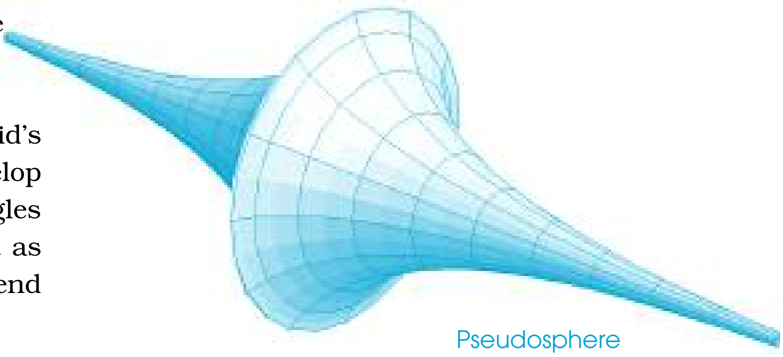
Georg Riemann
(1826–1866)

Riemann and Klein

The next example of what we could now call a 'non-euclidean' geometry was given by Riemann. A lecture he gave which was published in 1868, two years after his death, speaks of a 'spherical' geometry in which every line through a point P not on a line AB meets the line AB. Here, no parallels are possible. Also, in 1868, Eugenio Beltrami wrote a paper in which he puts forward a model called a 'pseudo-sphere'. The importance of this model is that it gave an example of the first four postulates holding but not the fifth. From this, it can be seen that non-euclidean geometry is just as consistent as euclidean geometry.

In 1871, Klein completed the ideas of non-euclidean geometry and gave the solid underpinnings to the subject. He shows that there are essentially three types of geometry:

- that proposed by Bolyai and Lobachevsky, where straight lines have two infinitely distant points,
- the Riemann 'spherical' geometry, where lines have no infinitely distant points, and
- Euclidean geometry, where for each line there are two coincident infinitely distant points.



Pseudosphere

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